BSDEs with terminal conditions that have bounded Malliavin derivative

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Abstract

We show existence and uniqueness of solutions to BSDEs of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s$$

in the case where the terminal condition ξ has bounded Malliavin derivative. The driver $f(s, y, z)$ is assumed to be Lipschitz continuous in y but only locally Lipschitz continuous in z. In particular, it can grow arbitrarily fast in z. If in addition to having bounded Malliavin derivative, ξ is bounded, the driver needs only be locally Lipschitz continuous in y. In the special case where the BSDE is Markovian, we obtain existence and uniqueness results for semilinear parabolic PDEs with non-Lipschitz nonlinearities.

Keywords: Backward stochastic differential equation, Malliavin derivative, semilinear parabolic PDE, Neumann boundary condition, Dirichlet boundary condition, viscosity solution.

1 Introduction

The aim of this paper is to show existence and uniqueness of solutions to BSDEs (backward stochastic differential equations) of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s \tag{1.1}$$

in the case where the terminal condition ξ has bounded Malliavin derivative.

$(W_t)_{0 \le t \le T}$ is an n-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and ξ is an $\mathcal{F}_T$-measurable random variable, where $(\mathcal{F}_t)_{0 \le t \le T}$ is the augmented filtration generated by W. The driver f is a function from $[0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}$ that is measurable with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^n)$, where $\mathcal{P}$ is the predictable sigma-algebra on $[0, T] \times \Omega$. As usual, we identify random variables that are equal $\mathbb{P}$-almost surely and accordingly, understand equalities and inequalities between them in the $\mathbb{P}$-almost sure sense. The Euclidean norm on $\mathbb{R}^d$ is denoted by $|.|$, and xy stands for $\sum_{i=1}^d x_i y_i$, $x, y \in \mathbb{R}^d$. We work with the following

Definition 1.1. *A solution of the BSDE* (1.1) *is a pair* $(Y_t, Z_t)_{0 \le t \le T}$ *of predictable processes taking values in* $\mathbb{R} \times \mathbb{R}^n$ *such that* $\int_0^T \left(|f(t, Y_t, Z_t)| + |Z_t|^2\right) dt < \infty$ *and* (1.1) *holds for all* $0 \le t \le T$.

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For $p \in [1, \infty]$, we denote

- $\mathbb{S}^p(\mathbb{R}^d) :=$ the space of $\mathbb{R}^d$-valued continuous adapted processes X satisfying

$$\|X\|_{\mathbb{S}^p} := \left\| \sup_{0 \le t \le T} |X_t| \right\|_{L^p} < \infty$$

  where processes X, Y are identified if $\|X - Y\|_{\mathbb{S}^p} = 0$.

- $\mathbb{H}^p(\mathbb{R}^d) :=$ the space of $\mathbb{R}^d$-valued predictable processes X satisfying

$$\|X\|_{\mathbb{H}^p} := \left\| \left(\int_0^T |X_t|^2 dt \right)^{1/2} \right\|_{L^p} < \infty \quad \text{if } p < \infty \text{ and}$$
$$\|X\|_{\mathbb{H}^\infty} := \operatorname*{ess\,sup}_{(t,\omega) \in [0,T] \times \Omega} |X_t(\omega)| < \infty \quad \text{if } p = \infty,$$

  where processes X, Y are identified if $\|X - Y\|_{\mathbb{H}^p} = 0$.

(f, ξ) are said to be p-standard parameters if they satisfy the following three conditions:

(S1) $\xi \in L^p(\mathcal{F}_T)$

(S2) $|f(t, y, z) - f(t, y', z')| \le L(|y - y'| + |z - z'|)$ for a constant $L \in \mathbb{R}_+$

(S3) $f(., 0, 0) \in \mathbb{H}^p(\mathbb{R})$.

It can be shown with a Picard iteration argument that for all $p \in (1, \infty)$, a BSDE of the form (1.1) with p-standard parameters has a unique solution (Y, Z) in $\mathbb{S}^p(\mathbb{R}) \times \mathbb{H}^p(\mathbb{R}^n)$; see Theorem 5.1 in El Karoui et al. [10]. Kobylanski [14] proved the existence of a unique solution in the case where f does not grow faster than quadratically in z and ξ is bounded. BSDEs with drivers of quadratic growth in z and unbounded terminal conditions have been studied by Briand and Hu [3, 4] as well as Delbaen et al. [8]. Delbaen et al. [7] showed that if the driver f only depends on z, is convex and has superquadratic growth, there exist bounded terminal conditions such that the BSDE (1.1) has no solution with bounded Y, and if the BSDE admits a solution with bounded Y, it has infinitely many of them. Moreover, they proved the existence of a solution for Markovian BSDEs when the terminal value is a bounded continuous function of the terminal value of a forward process. Richou [20] proved the existence of solutions to more general Markovian BSDEs in the case where f and ξ satisfy a local Lipschitz condition with respect to the underlying forward process. In Cheridito and Stadje [5] it is shown that BSDEs whose drivers are convex in z have unique solutions with bounded Z if f and ξ are Lipschitz continuous functionals of the path of the underlying Brownian motion.

In this paper f can grow arbitrarily fast in z, and we do not make Markov or convexity assumptions. On the other hand, we require f and ξ to be Malliavin differentiable with bounded Malliavin derivatives. We recall that $\mathcal{H} := L^2([0, T]; \mathbb{R}^n)$ is a Hilbert space with scalar product $\langle h_1, h_2 \rangle := \int_0^T h_1(t) h_2(t) dt$, and the mapping $h \mapsto \int_0^T h(t) dW_t$ is a Hilbert space isomorphism between $\mathcal{H}$ and the first Wiener chaos of W. The corresponding Malliavin derivative of a Malliavin differentiable random variable ξ is an n-dimensional stochastic process $D_t \xi$, $0 \le t \le T$, whose components we denote by $D_t^i \xi$, $i = 1, \dots, n$. The Sobolev space $\mathbb{D}^{1,2}$ is defined as the closure of the class of smooth random variables ξ with respect to the norm $\|\xi\|_{1,2} := \left(\mathbb{E} \left[\xi^2 + \int_0^T |D_t \xi|^2 dt \right] \right)^{1/2}$; see Nualart [16]. $\mathbb{L}_a^{1,2}(\mathbb{R}^d)$ denotes the space of $\mathbb{R}^d$-valued progressively measurable processes X satisfying

(i) $X_t \in (\mathbb{D}^{1,2})^d$ for almost all t

(ii) $(t,\omega) \mapsto DX_t(\omega) \in (L^2[0,T])^{n\times d}$ admits a progressively measurable version

(iii) $\|X\|^2_{\mathbb{L}_a^{1,2}} := \|X\|_{\mathbb{H}^2} + \left\| \left(\int_0^T \int_0^T |D_r X_t|^2 dr dt \right)^{1/2} \right\|_{L^2} < \infty,$

where processes X, Y are identified if $\|X - Y\|_{\mathbb{L}_a^{1,2}} = 0$.

Now consider the conditions:

(A1) The terminal condition ξ is in $\mathbb{D}^{1,2}$ and there exist constants $A_i \in \mathbb{R}_+$ such that $|D^i_t \xi| \leq A_i$ $dt \otimes d\mathbb{P}$-a.e. for all $i = 1, \ldots, n$.

(A2) There exist a constant $B \in \mathbb{R}_+$ and a nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t,y,z) - f(t,y',z)| \leq B|y - y'| \quad \text{and} \quad |f(t,y,z) - f(t,y,z')| \leq \rho(|z| \vee |z'|)|z - z'|$$

for all $t \in [0,T]$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^n$.

(A3) $f(.,0,0) \in \mathbb{H}^4(\mathbb{R})$ and there exist Borel-measurable functions $q_i : [0,T] \to \mathbb{R}_+$ satisfying $\int_0^T q_i^2(t) dt < \infty$ such that for every pair $(y,z) \in \mathbb{R} \times \mathbb{R}^n$ with

$$|z| \leq Q := \sqrt{\sum_{i=1}^n \left(A_i + \int_0^T q_i(t) e^{-B(T-t)} dt \right)^2} \, e^{BT},$$

one has $f(\cdot, y, z) \in \mathbb{L}_a^{1,2}(\mathbb{R})$ and $|D^i_r f(t,y,z)| \leq q_i(t)$ $dr \otimes d\mathbb{P}$-a.e. for all $i = 1, \ldots, n$.

(A4) For a.a. $r \in [0,T]$, there exists a non-negative process $K_{r.}$ in $\mathbb{H}^4(\mathbb{R})$ such that

$$\int_0^T \|K_{r.}\|^4_{\mathbb{H}^4} dr < \infty \quad \text{and} \quad |D_r f(t,y,z) - D_r f(t,y',z')| \leq K_{rt}(|y - y'| + |z - z'|)$$

for all $t \in [0,T]$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^n$ satisfying $|z|, |z'| \leq Q$.

Our main result is the following

Theorem 1.2. *If* (A1)–(A4) *hold, then the BSDE* (1.1) *has a unique solution* (Y,Z) *in* $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$*, and for all* $i = 1, \ldots, n$*,*

$$|Z^i_t| \leq \left(A_i + \int_t^T q_i(s) e^{-B(T-s)} ds \right) e^{B(T-t)} \quad dt \otimes d\mathbb{P}\text{-a.e.}$$

Remark 1.3. If for a.a. $r \in [0,T]$, the process $K_{r.}$ in (A4) is bounded, the condition $f(.,0,0) \in \mathbb{H}^4(\mathbb{R})$ can be dropped from (A3). Then the statement of Theorem 1.2 still holds, except that Y is in $\mathbb{S}^2(\mathbb{R})$ instead of $\mathbb{S}^4(\mathbb{R})$. This is due to the fact that in this case, $f(.,0,0) \in \mathbb{H}^4(\mathbb{R})$ is not needed in Proposition 2.1 below; see Remark 2.3.

In the next corollary, we assume that the terminal condition ξ is bounded and has bounded Malliavin derivative. This allows us to relax some of the assumptions of Theorem 1.2 on the driver f. The precise conditions we need are the following:

(B1) ξ satisfies (A1) and there exists a constant $C \in \mathbb{R}_+$ such that $|\xi| \leq C$.

(B2) There exist constants $B, D \in \mathbb{R}_+$ and a nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\begin{aligned} &|f(t,y,z) - f(t,y',z)| \leq B|y-y'| \\ &|f(t,y,z) - f(t,y,z')| \leq \rho(|z| \vee |z'|)|z-z'| \\ &|f(t,y,z)| \leq D(1+|y|) + \rho(|z|)|z| \end{aligned}$$

for all $t \in [0,T]$, $y, y' \in \mathbb{R}$ with $|y|, |y'| \leq R := (C+1)e^{DT} - 1$ and all $z, z' \in \mathbb{R}^n$.

(B3) Condition (A3) holds for all $(y,z) \in \mathbb{R} \times \mathbb{R}^n$ such that $|y| \leq R$ and $|z| \leq Q$.

(B4) Condition (A4) holds for all $t \in [0,T]$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^n$ such that $|y|, |y'| \leq R$ and $|z|, |z'| \leq Q$.

Corollary 1.4. *Assume* (B1)–(B4). *Then the BSDE* (1.1) *has a unique solution* (Y,Z) *in* $\mathbb{S}^\infty(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$*, and*

$$\begin{aligned} &|Y_t| \leq (C+1)e^{D(T-t)} - 1 \quad \textit{for all } t \in [0,T] \\ &|Z^i_t| \leq \left(A_i + \int_t^T q_i(s) e^{-B(T-s)} ds \right) e^{B(T-t)} \quad dt \otimes d\mathbb{P}\textit{-a.e.} \quad \textit{for all } i = 1, \dots, n. \end{aligned}$$

Theorem 1.2 and Corollary 1.4 are proved in Section 2. In Section 3 we show that every terminal condition that is Lipschitz in the underlying Brownian motion has a bounded Malliavin derivative. On the other hand, we give an example of a terminal condition with bounded Malliavin derivative that is not Lipschitz in the underlying Brownian motion. This shows that condition (A1) is weaker than Lipschitz continuity in the underlying Brownian motion. In Sections 4–6 we generalize results on the relation between Markovian BSDEs and semilinear parabolic PDEs to the case of non-Lipschitz nonlinearities. In Section 4 we study Markovian BSDEs based on forward processes following standard diffusion dynamics and related PDEs for functions $u : [0,T] \times \mathbb{R}^m \to \mathbb{R}$. Theorem 1.2 and Corollary 1.4 will allow us to extend results of Amour and Ben-Artzi [1] and Gilding et al. [11] on the existence of solutions to nonlinear heat equations. Section 5 is devoted to BSDEs with random terminal times and parabolic PDEs with Dirichlet boundary conditions. Finally, Section 6 discusses BSDEs based on reflected forward processes and their relation to parabolic PDEs with Neumann boundary conditions.

2 Proof of Theorem 1.2 and Corollary 1.4

In a first step we need the following stronger versions of conditions (A2)–(A4):

(A2') $f(t,y,z)$ is continuously differentiable in (y,z) and there exist constants $B, \rho \in \mathbb{R}_+$ such that

$$|\partial_y f(t,y,z)| \leq B, \quad |\partial_z f(t,y,z)| \leq \rho$$

for all $t \in [0,T]$, $y \in \mathbb{R}$ and $z \in \mathbb{R}^n$.

(A3') Condition (A3) holds for all $(y,z) \in \mathbb{R} \times \mathbb{R}^n$.

(A4') Condition (A4) holds for all $t \in [0,T]$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^n$.

Proposition 2.1. *If* (A1), (A2'), (A3'), (A4') *are satisfied, then the BSDE* (1.1) *has a unique solution* (Y, Z) *in* $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^4(\mathbb{R}^n)$*, and*

$$|Z_t^i| \leq \left(A_i + \int_t^T q_i(s) e^{-B(T-s)} ds \right) e^{B(T-t)} \quad dt \otimes d\mathbb{P}\text{-a.e.} \tag{2.1}$$

Proof. By Lemma 2.2 below, condition (A1) implies $\mathbb{E}|\xi|^p < \infty$ for all $p \in \mathbb{R}_+$. So it follows from Theorem 5.1 and Proposition 5.3 of El Karoui et al. [10] that the BSDE (1.1) has a unique solution (Y, Z) in $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^4(\mathbb{R}^n)$, and $(Y, Z) \in \mathbb{L}_a^{1,2}(\mathbb{R}^{n+1})$. Moreover, for all $i = 1, \dots, n$,

$$(D_r^i Y_t, D_r^i Z_t) = (U_t^r, V_t^r) \;\; dr \otimes dt \otimes d\mathbb{P}\text{-a.e.} \quad \text{and} \quad Z_t^i = U_t^t \;\; dt \otimes d\mathbb{P}\text{-a.e.},$$

where

$$U_t^r = 0, \; V_t^r = 0, \quad 0 \leq t < r \leq T,$$

and for each fixed r, $(U_t^r, V_t^r)_{r \leq t \leq T}$ is the unique pair in $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^n)$ solving the BSDE

$$U_t^r = D_r^i \xi + \int_t^T [\partial_y f(s, Y_s, Z_s) U_s^r + \partial_z f(s, Y_s, Z_s) V_s^r + D_r^i f(s, Y_s, Z_s)] ds - \int_t^T V_s^r dW_s. \tag{2.2}$$

Since (2.2) and the two BSDEs

$$\overline{U}_t = A_i + \int_t^T \left(B|\overline{U}_s| + \rho|\overline{V}_s| + q_i(s) \right) ds - \int_t^T \overline{V}_s dW_s \tag{2.3}$$

$$\underline{U}_t = -A_i - \int_t^T \left(B|\underline{U}_s| + \rho|\underline{V}_s| + q_i(s) \right) ds - \int_t^T \underline{V}_s dW_s \tag{2.4}$$

have 2-standard parameters, one obtains from the comparison result, Theorem 2.2 in El Karoui et al. [10], that $\underline{U}_t \leq U_t^r \leq \overline{U}_t$ for all $t \in [0, T]$. But the solutions to (2.3) and (2.4) are given by

$$\overline{U}_t = -\underline{U}_t = \left(A_i + \int_t^T q_i(s) e^{-B(T-s)} ds \right) e^{B(T-t)}, \quad \overline{V}_t = \underline{V}_t = 0.$$

This shows (2.1). □

Lemma 2.2. *If* ξ *satisfies* (A1)*, then* $\mathbb{E}|\xi|^p < \infty$ *for all* $p \in [1, \infty)$.

Proof. If ξ satisfies (A1), it is square-integrable. By the Clark–Ocone formula, one can represent ξ as $\xi = \mathbb{E}[\xi] + \int_0^T \mathbb{E}[D_t \xi | \mathcal{F}_t] dW_t$. Applying the Burkholder–Davis–Gundy inequality to the martingale $M_t = \int_0^t \mathbb{E}[D_t \xi | \mathcal{F}_s] dW_s$, one obtains a constant $c_p \in \mathbb{R}_+$ such that

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |M_t|^p \right] \leq c_p \mathbb{E}\left[\left(\int_0^T |\mathbb{E}[D_t \xi | \mathcal{F}_t]|^2 dt \right)^{p/2} \right] < \infty,$$

which proves the lemma. □

Remark 2.3. If for a.a. $r \in [0, T]$, the process $K_{r.}$ in (A4') is bounded, Proposition 2.1 still holds if the condition $f(., 0, 0) \in \mathbb{H}^4(\mathbb{R})$ is dropped from (A3') except that then, (Y, Z) is in $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^n)$ and not necessarily in $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^4(\mathbb{R}^n)$. This is true because in this case, the proof of Proposition 5.3 in El Karoui et al. [10] still works without the assumption $f(., 0, 0) \in \mathbb{H}^4(\mathbb{R})$ with the difference that it yields a solution (Y, Z) of the BSDE (1.1) in $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^n)$ instead of $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^4(\mathbb{R}^n)$.

To derive Theorem 1.2 from Proposition 2.1, we need the following result, which is Proposition 5.1 of El Karoui et al. [10] in the special case of a Brownian filtration and $p = 2$.

Proposition 2.4. (El Karoui et al., 1997) *For every $L \in \mathbb{R}_+$ there exist constants $\mu, \nu > 0$ satisfying the following: If $T \leq \mu$, then for all 2-standard parameters (f^i, ξ^i), $i = 1, 2$, such that f^1 fulfills the Lipschitz condition* (S2) *with Lipschitz constant L, the BSDE solutions (Y^i, Z^i) corresponding to (f^i, ξ^i) satisfy*

$$\left\|Y^1 - Y^2\right\|^2_{\mathbb{S}^2} + \left\|Z^1 - Z^2\right\|^2_{\mathbb{H}^2} \leq \nu\, \mathbb{E}\left[\left|\xi^1 - \xi^2\right|^2 + \int_0^T \left(f^1(t, Y^2_t, Z^2_t) - f^2(t, Y^2_t, Z^2_t)\right)^2 dt\right].$$

Proof of Theorem 1.2
Define

$$\hat{f}(t, y, z) = \begin{cases} f(t, y, z) & \text{if } |z| \leq Q \\ f(t, y, Qz/|z|) & \text{if } |z| > Q \end{cases}.$$

Then $(\hat{f}, \xi)$ are 4-standard parameters. So the corresponding BSDE has a unique solution (Y, Z) in $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^4(\mathbb{R}^n)$. Denote $x = (y, z) \in \mathbb{R}^{n+1}$ and let $\beta \in C_c^\infty(\mathbb{R}^{n+1})$ be the mollifier

$$\beta(x) := \begin{cases} \lambda \exp\left(-\frac{1}{1-|x|^2}\right) & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases},$$

where the constant $\lambda \in \mathbb{R}_+$ is chosen so that $\int_{\mathbb{R}^{n+1}} \beta(x)dx = 1$. Set $\beta^m(x) := m^{n+1}\beta(mx)$, $m \in \mathbb{N} \setminus \{0\}$, and define

$$f^m(t, \omega, x) := \int_{\mathbb{R}^{n+1}} \hat{f}(t, \omega, x')\beta^m(x - x')dx'.$$

Then all f^m satisfy (A2')–(A4'). Therefore, one obtains from Proposition 2.1 that there exist unique solutions (Y^m, Z^m) in $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^4(\mathbb{R})$ to the BSDEs corresponding to (f^m, ξ), and $|Z^{m,i}_t| \leq a_i(t) := (A_i + \int_t^T q_i(s)e^{-B(T-s)}ds)e^{B(T-t)}$. Since $\hat{f}$ satisfies the Lipschitz condition (S2) for some constant $L \in \mathbb{R}_+$, one can choose constants $\mu, \nu > 0$ such that the statement of Proposition 2.4 holds. This gives

$$\|Y - Y^m\|^2_{\mathbb{S}^2,[T-\mu,T]} + \|Z - Z^m\|^2_{\mathbb{H}^2,[T-\mu,T]} \leq \nu\, \mathbb{E}\left[\int_{T-\mu}^T \left(\hat{f}(t, Y^m_t, Z^m_t) - f^m(t, Y^m_t, Z^m_t)\right)^2 dt\right].$$

Since $\left|\hat{f} - f^m\right| \to 0$ uniformly in (t, ω, y, z) as $m \to \infty$, one obtains $\mathbb{E}\left[(Y_{T-\mu} - Y^m_{T-\mu})^2\right] \to 0$ and $|Z^i_t| \leq a_i(t)$ for $T - \mu \leq t \leq T$. Proposition 2.4 applied on the interval $[T - 2\mu, T - \mu]$ yields

$$\begin{aligned} &\|Y - Y^m\|^2_{\mathbb{S}^2,[T-2\mu,T-\mu]} + \|Z - Z^m\|^2_{\mathbb{H}^2,[T-2\mu,T-\mu]} \\ \leq\ & \nu\, \mathbb{E}\left[(Y_{T-\mu} - Y^m_{T-\mu})^2 + \int_{T-2\mu}^{T-\mu} \left(\hat{f}(t, Y^m_t, Z^m_t) - f^m(t, Y^m_t, Z^m_t)\right)^2 dt\right]. \end{aligned}$$

So $\mathbb{E}\left[(Y_{T-2\mu} - Y^m_{T-2\mu})^2\right] \to 0$ and $|Z^i_t| \leq a_i(t)$ for $T - 2\mu \leq t \leq T - \mu$. By repeating this argument, one gets $|Z^i(t)| \leq a_i(t)$ for all $t \in [0, T]$. It follows that (Y, Z) is also a solution of the BSDE (1.1) with parameters (f, ξ).

Finally, if $(\tilde{Y}, \tilde{Z})$ is another solution in $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$ corresponding to (f, ξ), it must be equal to (Y, Z) since both solve the BSDE (1.1) with a 4-standard driver $\tilde{f}$ that coincides with f for $|z| \leq \tilde{Q}$, where $\tilde{Q} \in \mathbb{R}_+$ is a bound on Z and $\tilde{Z}$. □

Proof of Corollary 1.4
Consider the following three BSDEs

$$Y_t = \xi + \int_t^T \hat{f}(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s \tag{2.5}$$

$$\overline{Y}_t = C + \int_t^T \overline{f}(s, \overline{Y}_s, \overline{Z}_s)ds - \int_t^T \overline{Z}_s dW_s \tag{2.6}$$

$$\underline{Y}_t = -C + \int_t^T \underline{f}(s, \underline{Y}_s, \underline{Z}_s)ds - \int_t^T \underline{Z}_s dW_s, \tag{2.7}$$

where $\hat{f}(t, y, z) := f(t, \tilde{y}, \tilde{z})$ for

$$\tilde{y} := \begin{cases} y & \text{if } |y| \leq R \\ Ry/|y| & \text{if } |y| > R \end{cases} \quad \text{and} \quad \tilde{z} := \begin{cases} z & \text{if } |z| \leq Q \\ Qz/|z| & \text{if } |z| > Q \end{cases},$$

$\overline{f}(t, y, z) := D(1 + |y|) + \rho(Q)|z|$ and $\underline{f}(t, y, z) := -\overline{f}(t, y, z)$. $\hat{f}$ satisfies (A2)–(A4) and has the following two properties:

1) $\hat{f}(t, y, z) = f(t, y, z)$ for all (t, y, z) such that $|y| \leq R$ and $|z| \leq Q$

2) $\underline{f}(t, y, z) \leq \hat{f}(t, y, z) \leq \overline{f}(t, y, z)$ for all (t, y, z).

It follows from Theorem 1.2 that (2.5) has a unique solution (Y, Z) in $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$, and

$$|Z_t^i| \leq \left(A_i + \int_t^T q_i(s) e^{-B(T-s)} ds \right) e^{B(T-t)}.$$

Moreover, one obtains from Theorem 2.2 in El Karoui et al. [10] that

$$\underline{Y}_t \leq Y_t \leq \overline{Y}_t, \quad 0 \leq t \leq T,$$

and it can easily be checked that

$$\overline{Y}_t = -\underline{Y}_t = (C + 1)e^{D(T-t)} - 1, \quad \overline{Z}_t = \underline{Z}_t = 0.$$

This gives $|Y_t| \leq (C + 1)e^{D(T-t)} - 1 \leq R$. So (Y, Z) solves the BSDE (1.1) with parameters (f, ξ).

To conclude the proof, assume that $(\tilde{Y}, \tilde{Z})$ is another solution in $\mathbb{S}^\infty(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$. Let $\tilde{Q} \in \mathbb{R}_+$ be a bound on $\tilde{Z}$ and assume

$$t^* := \sup \left\{ s \in [0, T] : \mathbb{P}[|\tilde{Y}_s| \geq R] > 0 \right\} > 0.$$

On $[t^*, T]$, $\tilde{Y}$ is bounded by R, and hence, $(\tilde{Y}, \tilde{Z})$ is equal to (Y, Z) since both solve the BSDE (1.1) with a 4-standard driver $\tilde{f}$ that coincides with f for $|y| \leq R$ and $|z| \leq Q \vee \tilde{Q}$. In particular, $|\tilde{Y}_{t^*}| \leq (C + 1)e^{D(T-t^*)} - 1 < R$. It follows that there exists an $\varepsilon > 0$ such that

$$|\tilde{Y}_t| = |\mathbb{E}_t \tilde{Y}_{t^*} + \int_t^{t^*} \mathbb{E}_t f(s, \tilde{Y}_s, \tilde{Z}_s)ds| \leq (C + 1)e^{D(T-t^*)} - 1 + (t^* - t)[D(1 + R) + \rho(\tilde{Q})\tilde{Q}] < R$$

for all $t \in [t^* - \varepsilon, t^*]$, a contradiction to the definition of t^*. This shows that $t^* = 0$ and $(\tilde{Y}, \tilde{Z}) = (Y, Z)$. $\square$

3 Lipschitz continuity and Bounded Malliavin derivatives

In this section we show that terminal conditions ξ that are Lipschitz continuous in the underlying Brownian motion W are Malliavin differentiable with bounded Malliavin derivative. On the other hand, we give an example of a terminal condition with bounded Malliavin derivative which is not Lipschitz continuous in W. This shows that condition (A1) is more general than Lipschitz continuity in W.

Definition 3.1. *We denote the space of all continuous functions from $[0,T]$ to $\mathbb{R}^n$ starting from 0 by $C_0^n[0,T]$ and call a random variable ξ Lipschitz continuous in the Brownian motion W with constants $A_1,\dots,A_n \in \mathbb{R}_+$ if $\xi = \varphi(W)$ for a function $\varphi : C_0^n[0,T] \to \mathbb{R}$ satisfying*

$$|\varphi(v) - \varphi(w)| \le \sum_{i=1}^n A_i \sup_{0\le t\le T} |v^i(t) - w^i(t)|. \tag{3.1}$$

Proposition 3.2. *Let ξ be Lipschitz continuous in W with constants $A_1,\dots,A_n \in \mathbb{R}_+$. Then $\xi \in \mathbb{D}^{1,2}$ and $|D_t^i \xi| \le A_i$ $dt \otimes d\mathbb{P}$-a.e. for all $i = 1,\dots,n$.*

Proof. Assume ξ is of the form $\varphi(W)$ for a function φ satisfying (3.1). For $m \in \mathbb{N}$, set $t_j^m := jT/m$, $j = 0,\dots,m$, and define the mapping $l^m : \left\{ x = (x_j)_{j=1}^m : x_j \in \mathbb{R}^n \right\} \to C_0^n[0,T]$ by

$$l_0^m(x) := 0 \quad \text{and} \quad l_t^m(x) := x_1 + \dots + x_{j-1} + \frac{t - t_{j-1}^m}{T/m} x_j \quad \text{for } t_{j-1}^m < t \le t_j^m.$$

Set $\xi^m := \varphi \circ l^m(\Delta W_{t_1^m},\dots,\Delta W_{t_m^m})$. For every $p \in [2,\infty)$, there exists a constant $b_p \in \mathbb{R}_+$ such that

$$\begin{aligned}
&\mathbb{E}|\xi - \xi^m|^p \le b_p \,\mathbb{E} \sup_{0\le t\le T} |W_t^1 - l_t^{m,1}(\Delta W_{t_1^m},\dots,\Delta W_{t_m^m})|^p \\
\le\;& b_p \,\mathbb{E} \max_{j=1,\dots,m} \sup_{t_{j-1}^m < t \le t_j^m} \left| W_t^1 - W_{t_{j-1}^m}^1 - \frac{t - t_{j-1}^m}{T/m} \Delta W_{t_j^m}^1 \right|^p \\
\le\;& b_p m \mathbb{E} \sup_{0<t\le T/m} \left| W_t^1 - \frac{t W_{T/m}^1}{T/m} \right|^p,
\end{aligned}$$

where for the last inequality, we used that W has stationary increments. It follows that

$$\begin{aligned}
&\|\xi - \xi^m\|_p \le (b_p m)^{1/p} \left\| \sup_{0<t\le T/m} \left| W_t^1 - \frac{t W_{T/m}^1}{T/m} \right| \right\|_p \le (b_p m)^{1/p} \left(\left\| \sup_{0<t\le T/m} |W_t^1| \right\|_p + \left\| W_{T/m}^1 \right\|_p \right) \\
\le\;& (b_p m)^{1/p} c_p \left\| W_{T/m}^1 \right\|_p \le (b_p m)^{1/p} d_p \sqrt{T/m},
\end{aligned}$$

where c_p and d_p are constants depending on p, and the second inequality follows from Doob's maximal inequality. For $p > 2$ the last term goes to 0 as $m \to \infty$. This shows that $\xi^m \to \xi$ in L^p for all $p \in (2,\infty)$ and therefore also in L^2.

Note that for $x, y \in \mathbb{R}^{mn}$,

$$|\varphi \circ l^m(x) - \varphi \circ l^m(y)| \le \sum_{i,j} A_i |x_j^i - y_j^i|. \tag{3.2}$$

Let $\beta \in C_c^\infty(\mathbb{R}^{mn})$ be the mollifier

$$\beta(x) := \begin{cases} \lambda \exp\left(-\frac{1}{1-|x|^2}\right) & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases},$$

where λ is a constant so that $\int_{\mathbb{R}^{mn}} \beta(x)dx = 1$. Set $\beta^m(x) := m^{mn}\beta(mx)$ and define

$$\varphi^m(x) := \int_{\mathbb{R}^{mn}} \varphi \circ l^m(y)\beta^m(x-y)dy, \quad \tilde{\xi}^m := \varphi^m(\Delta W_{t_1^m}, \dots, \Delta W_{t_m^m}).$$

By Proposition 1.2.3 of Nualart [16], one has

$$D^i \tilde{\xi}^m = \sum_{j=1}^m \frac{\partial}{\partial x_j^i} \varphi^m(\Delta W_{t_1^m}, \dots, \Delta W_{t_m^m}) 1_{(t_{j-1}^m, t_j^m]}.$$

But it follows from (3.2) that $\left|\frac{\partial}{\partial x_j^i}\varphi^m(x)\right| \le A_i$ for all i, j. So $|D_t^i \tilde{\xi}^m| \le A_i \; dt \otimes d\mathbb{P}$-a.e. Moreover $\tilde{\xi}^m \to \xi$ in L^2. Hence, one obtains from Lemma 1.2.3 of Nualart [16] that ξ is in $\mathbb{D}^{1,2}$ and $D\tilde{\xi}^m \to D\xi$ in the weak topology of $L^2(\Omega; H)$. This implies that $|D_t^i \xi| \le A_i \; dt \otimes d\mathbb{P}$-a.e. $\square$

In the following example we construct a random variable with bounded Malliavin derivative that is not Lipschitz in the underlying Brownian motion.

Example 3.3. Assume $T = n = 1$. Define

$$g(t) := \sum_{k=1}^\infty (-1)^{k-1} 2^k 1_{\{1-2^{1-k} < t \le 1-2^{-k}\}}, \quad h(t) := \int_0^t g(s)ds,$$

and set

$$\xi := \int_0^1 h(t)dW_t.$$

Then $\xi \in \mathbb{D}^{1,2}$ and $D\xi = h$ is bounded by 1.

On the other hand, it follows from integration by parts that

$$\int_0^{1-2^{-2k}} h(t)dW_t = -\int_0^{1-2^{-2k}} g(t)W_t dt \quad \text{for all } k \ge 1.$$

Therefore,

$$\xi = -\lim_{k\to\infty} \int_0^{1-2^{-2k}} g(t)W_t dt,$$

which shows that ξ cannot be of the form $\xi = \varphi(W)$ for a Lipschitz continuous function $\varphi : C_0[0,1] \to \mathbb{R}$.

4 Markovian BSDEs and semilinear parabolic PDEs

For $(t,x) \in [0,T] \times \mathbb{R}^m$, we consider an SDE of the form

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) dr + \int_t^s \sigma(r) dW_r \quad t \le s \le T, \tag{4.1}$$

where $b : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ and $\sigma : [0,T] \to \mathbb{R}^{m \times n}$ are Borel measurable functions for which there exist constants $E, F \in \mathbb{R}_+$ such that for all $t \in [0,T]$, $x, x' \in \mathbb{R}^m$ and i, j,

$$|\sigma_{ij}(t)| \le E \tag{4.2}$$

$$|b_i(t,x)| \le F(1 + \max_k |x_k|) \tag{4.3}$$

$$|b_i(t,x) - b_i(t,x')| \le F \max_k |x_k - x'_k|. \tag{4.4}$$

Denote $W_s^t := W_s - W_t$, $s \in [t,T]$, and let $(\mathcal{F}_s^t)_{s \in [t,T]}$ be the filtration generated by W^t. By $\mathbb{S}_t^p(\mathbb{R}^d)$ we denote the space of all $\mathbb{R}^d$-valued continuous $(\mathcal{F}_s^t)$-adapted processes with finite $\mathbb{S}^p$-norm on $[t,T]$, and by $\mathbb{H}_t^p(\mathbb{R}^d)$ the space of all $\mathbb{R}^d$-valued $(\mathcal{F}_s^t)$-predictable processes with finite $\mathbb{H}^p$-norm on $[t,T]$. Analogously, we denote by $\mathbb{D}_t^{1,2}$ and $\mathbb{L}_{a,t}^{1,2}$ the spaces $\mathbb{D}^{1,2}$ and $\mathbb{L}_a^{1,2}$ with respect to $(W_s^t)_{s \in [t,T]}$.

Under (4.2)–(4.4) the SDE (4.1) has a unique strong solution in $\mathbb{S}_t^2(\mathbb{R}^m)$; see for instance, Karatzas and Shreve [13]. A Markovian BSDE based on $X^{t,x}$ is of the form

$$Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T Z_r^{t,x} dW_r \tag{4.5}$$

for measurable functions $g : [0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^m \to \mathbb{R}$.

It is well-known that if g is sufficiently regular in (r,x) and Lipschitz in (y,z), $u(t,x) = Y_t^{t,x}$ is a viscosity solution of the parabolic PDE with terminal condition

$$u_t(t,x) + \mathcal{L}_{(t,x)} u(t,x) + g(t,x,u(t,x), \nabla u \sigma(t,x)) = 0, \quad u(T,x) = h(x), \tag{4.6}$$

where

$$\mathcal{L}_{(t,x)} := \frac{1}{2} \sum_{i,j} (\sigma \sigma^T)_{ij}(t) \partial_{x_i} \partial_{x_j} + \sum_i b_i(t,x) \partial_{x_i};$$

see El Karoui et al. [10]. Since Theorem 1.2 and Corollary 1.4 give bounds on solutions of BSDEs, we can generalize this relationship between BSDEs and PDEs to the case where g is non-Lipschitz in (y,z). To do that we require g and h to satisfy the following conditions:

(C1) There exists a constant $A \in \mathbb{R}_+$ such that $|h(x) - h(x')| \le A \max_i |x_i - x'_i|$ for all $x, x' \in \mathbb{R}^m$.

(C2) There exist a constant $B \in \mathbb{R}_+$ and a nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(t,x,y,z) - g(t,x,y',z)| \le B|y - y'| \text{ and } |g(t,x,y,z) - g(t,x,y,z')| \le \rho\left(|z| \vee |z'|\right) |z - z'|$$

for all $t \in [0,T]$, $x \in \mathbb{R}^m$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^n$.

(C3) $\int_0^T g(t,0,0,0)^2 dt < \infty$ and there exists a constant $G \in \mathbb{R}_+$ such that for every pair $(y,z) \in \mathbb{R} \times \mathbb{R}^n$ with

$$|z| \le N := \sqrt{n} \left(A + \frac{1 - e^{-BT}}{B} G \right) E e^{(B+F)T}$$

one has $|g(t,x,y,z) - g(t,x',y,z)| \le G \max_i |x_i - x'_i|$ for all $t \in [0,T]$ and $x, x' \in \mathbb{R}^m$.

(C4) There exists a constant $H \in \mathbb{R}_+$ such that

$$\left|g(t,x,y,z) - g(t,x',y,z) - g(t,x,y',z') + g(t,x',y',z')\right| \le H \max_i |x_i - x'_i| \left(|y-y'| + |z-z'|\right)$$

for all $t \in [0,T]$, $x, x' \in \mathbb{R}^m$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^n$ with $|z|, |z'| \le N$.

Proposition 4.1. *Assume* (C1)–(C4). *Then for every* $(t,x) \in [0,T] \times \mathbb{R}^m$, *the Markovian BSDE* (4.5) *has a unique solution* $(Y^{t,x}, Z^{t,x})$ *in* $\mathbb{S}^2_t(\mathbb{R}) \times \mathbb{H}^\infty_t(\mathbb{R}^n)$, *and*

$$|Z^{t,x,i}_s| \le \left(A + \frac{1-e^{-B(T-s)}}{B} G\right) E e^{B(T-s)} e^{F(T-t)} \quad ds \otimes d\mathbb{P}\text{-a.e.} \quad \text{for all } i = 1, \dots, n.$$

Proof. If we can show that the BSDE (4.5) satisfies (A1) with $A_i = AEe^{F(T-t)}$, (A2), (A3) with $q_i \equiv GEe^{F(T-t)}$ but without $f(.,0,0) \in \mathbb{H}^4(\mathbb{R})$ and (A4) with a constant K, then the proposition follows from Theorem 1.2 and Remark 1.3.

(A2) is a direct consequence of (C2). By Lemma 4.2 below, $X^{t,x}_s$ is in $(\mathbb{D}^{1,2}_t)^m$ for all $t \le s \le T$ and $|D^i_r X^{t,x,j}_s| \le Ee^{F(T-t)}$ $dr \otimes d\mathbb{P}$-a.e. for all i and j. It follows from the Lipschitz condition (C1) and Proposition 1.2.4 of Nualart [16] that $h(X^{t,x}_T)$ is in $\mathbb{D}^{1,2}_t$ and for all $i = 1, \dots, n$, there exists an m-dimensional random vector Λ satisfying

$$D^i_r h(X^{t,x}_T) = \sum_{j=1}^m \Lambda^j D^i_r X^{t,x,j}_T \quad \text{and} \quad \sum_{j=1}^m |\Lambda^j| \le A.$$

This shows that the terminal condition $\xi = h(X^{t,x}_T)$ satisfies (A1) with $A_i = AEe^{F(T-t)}$. Analogously, it follows from (C3) that for every pair (y,z) such that $|z| \le N$, $g(., X^{t,x}_., y, z)$ belongs to $\mathbb{L}^{1,2}_{a,t}$ and $|D^i_r g(s, X^{t,x}_s, y, z)| \le GEe^{F(T-t)}$. So (A3) holds with $q_i \equiv GEe^{F(T-t)}$. The same argument applied to

$$\tilde{g}(s,x,y,y',z,z') = g(s,x,y,z) - g(s,x,y',z')$$

gives $|D^i_r g(s, X^{t,x}_s, y, z) - D^i_r g(s, X^{t,x}_s, y', z')| \le HEe^{F(T-t)}(|y-y'| + |z-z'|)$ for all $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^n$ with $|z|, |z'| \le N$. This shows that (A4) holds with a constant K. $\square$

Lemma 4.2. *For all* $0 \le t \le s \le T$ *and* $x \in \mathbb{R}^m$, $X^{t,x}_s$ *is in* $(\mathbb{D}^{1,2}_t)^m$ *and*

$$|D^i_r X^{t,x,j}_s| \le Ee^{F(T-t)} \quad dr \otimes d\mathbb{P}\text{-a.e.} \quad \text{for all } i = 1, \dots, n \text{ and } j = 1, \dots, m.$$

Proof. It follows from Theorem 2.2.1 of Nualart [16] that $X^{t,x}_s$ is in $(\mathbb{D}^{1,2}_t)^m$. Moreover, one obtains from the Lipschitz condition (4.4) and Proposition 1.2.4 of Nualart [16] that there exists an $\mathbb{R}^{m \times m}$-valued process Λ such that

$$D^i b_j(s, X^{t,x}_s) = \sum_{l=1}^m \Lambda^{jl}_s D^i X^{t,x,l}_s \quad \text{and} \quad \sum_{l=1}^m |\Lambda^{jl}_s| \le F.$$

It follows that $\int_t^s b_j(u, X^{t,x}_u) du \in \mathbb{D}^{1,2}_t$ with

$$\left| D^i_r \int_t^s b_j(u, X^{t,x}_u) du \right| \le \int_t^s \left| D^i_r b_j(u, X^{t,x}_u) \right| du \le F \int_t^s \max_l |D^i_r X^{t,x,l}_u| du.$$

Moreover, $|D^i \int_t^s \sum_{l=1}^n \sigma_{jl}(u) dW_u^l| = |\sigma_{ji} 1_{[t,s]}| \leq E$. Therefore,

$$\max_j |D_r^i X_s^{t,x,j}| \leq E + F \int_t^s \max_j |D_r^i X_u^{t,x,j}| du,$$

and one obtains from Gronwall's lemma that $|D_r^i X_s^{t,x,j}| \leq E e^{F(s-t)}$ $dr \otimes d\mathbb{P}$-a.e. $\square$

If the function h is bounded, one can relax some of the assumptions of Propositon 4.1 on g as follows:

(D1) The function h satisfies (C1) and is bounded by a constant $C \in \mathbb{R}_+$.

(D2) There exist constants $B, D \in \mathbb{R}_+$ and a nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\begin{aligned} &|g(t,x,y,z) - g(t,x,y',z)| \leq B|y-y'| \\ &|g(t,x,y,z) - g(t,x,y,z')| \leq \rho(|z| \vee |z'|)|z-z'| \\ &|g(t,x,y,z)| \leq D(1+|y|) + \rho(|z|)|z| \end{aligned}$$

for all $t \in [0,T]$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}$ with $|y|, |y'| \leq R := (C+1)e^{DT} - 1$ and all $z, z' \in \mathbb{R}^n$.

(D3) Condition (C3) holds for all $(y,z) \in \mathbb{R} \times \mathbb{R}^n$ such that $|y| \leq R$ and $|z| \leq N$.

(D4) Condition (C4) holds for all $t \in [0,T]$, $x, x' \in \mathbb{R}^m$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^n$ such that $|y|, |y'| \leq R$ and $|z|, |z'| \leq N$.

Proposition 4.3. *Assume* (D1)–(D4). *Then for all $(t,x) \in [0,T] \times \mathbb{R}^m$, the Markovian BSDE* (4.5) *has a unique solution $(Y^{t,x}, Z^{t,x})$ in $\mathbb{S}_t^\infty(\mathbb{R}) \times \mathbb{H}_t^\infty(\mathbb{R}^n)$, and*

$$\begin{aligned} &|Y_s^{t,x}| \leq (C+1)e^{D(T-s)} - 1 \quad \text{for all } s \in [t,T] \\ &|Z_s^{t,x,i}| \leq \left(A + \frac{1 - e^{-B(T-s)}}{B} G\right) E e^{B(T-s)} e^{F(T-t)} \quad ds \otimes d\mathbb{P}\text{-a.e.} \quad \text{for all } i = 1, \dots, n. \end{aligned}$$

Proof. (D1)–(D4) imply (B1)–(B4). Therefore, the proposition follows from Corollary 1.4 like Proposition 4.1 follows from Theorem 1.2. $\square$

Corollary 4.4. *If the assumptions of Proposition 4.1 or Proposition 4.3 hold, then the PDE* (4.6) *has a viscosity solution u such that for all $(t,x) \in [0,T] \times \mathbb{R}^m$, $u(s, X_s^{t,x}) = Y_s^{t,x}$, $t \leq s \leq T$, where $X^{t,x}$ and $Y^{t,x}$ are solutions of* (4.1) *and* (4.5)*, respectively.*

Proof. If the assumptions of Proposition 4.1 hold, the BSDE (4.5) has for all $(t,x) \in [0,T] \times \mathbb{R}^m$ a solution $(Y^{t,x}, Z^{t,x})$ such that $Z^{t,x}$ is bounded by N. So $(Y^{t,x}, Z^{t,x})$ also solves (4.5) if g is replaced by a function $\tilde{g}$ that agrees with g for $|z| \leq N$ and is Lipschitz in (x,y,z). It follows from Theorem 4.3 of Pardoux and Peng [17] that $u(t,x) := Y_t^{t,x}$ is a viscosity solution of (4.6) such that $u(s, X_s^{t,x}) = Y_s^{t,x}$, $t \leq s \leq T$.

Under the assumptions of Proposition 4.3, the BSDE (4.5) has a solution $(Y^{t,x}, Z^{t,x})$ such that $Y^{t,x}$ is bounded by $(C+1)e^{DT} - 1$ and $Z^{t,x}$ by N. Then $(Y^{t,x}, Z^{t,x})$ still solves (4.5) if g is replaced by a function $\tilde{g}$ that is Lipschitz in (x,y,z) and agrees with g for $|y| \leq (C+1)e^{DT} - 1$ and $|z| \leq N$. As above it follows that $u(t,x) := Y_t^{t,x}$ is a viscosity solution of (4.6) such that $u(s, X_s^{t,x}) = Y_s^{t,x}$, $t \leq s \leq T$. $\square$

Corollary 4.5. *Assume the conditions of Proposition 4.3 hold and set* $u(t,x) := Y_t^{t,x}$. *If for every* $L \in \mathbb{R}_+$, *there exists a constant* $\gamma_L \in \mathbb{R}$ *and a continuous function* $\delta_L : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\delta_L(0) = 0$ *such that*

$$\begin{aligned} g(t,x,y',v\sigma(t)) - g(t,x,y,v\sigma(t)) &\geq \gamma_L(y-y') \\ |g(t,x,y,v\sigma(t)) - g(t,x',y,v\sigma(t))| &\leq \delta_L(|x-x'|(1+|v|)) \end{aligned} \tag{4.7}$$

for all $(t,x,x') \in [0,T] \times \mathbb{R}^m \times \mathbb{R}^m$, $-L \leq y' \leq y \leq L$ *and* $v \in \mathbb{R}^m$, *then* u *is the unique bounded viscosity solution of the PDE* (4.6).

Proof. This follows from Section 4.2 of Ishii and Lions [12]. ☐

Under appropriate assumptions on the coefficients b, σ, g and h, the PDE (4.6) has a unique classical solution.

Corollary 4.6. *Assume* $\int_0^T g^2(t,0,0,0)dt < \infty$, b *only depends on* x, σ *is a constant and* b, g, h *are all* C^3 *in* (x,y,z). *Then one has the following:*

a) *If* (C1)–(C2) *hold and* b, g, h *have bounded derivatives of first, second and third order in* (x,y,z) *on the set* $\{(t,x,y,z) \in [0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n : |z| \leq N\}$, *then the PDE* (4.6) *has a unique solution* u *of class* $C^{1,2}$ *such that* $\nabla u\sigma$ *is bounded, and*

$$|\nabla u\sigma(t,x)| \leq \sqrt{n}\left(A + \frac{1-e^{-B(T-t)}}{B}G\right)Ee^{(B+F)(T-t)} \quad \textit{for all } (t,x) \in [0,T] \times \mathbb{R}^m.$$

b) *If* (D1)–(D2) *hold and* b, g, h *have bounded derivatives of first, second and third order in* (x,y,z) *on the set* $\left\{(t,x,y,z) \in [0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n : |y| \leq (C+1)e^{DT} - 1, |z| \leq N\right\}$, *then* (4.6) *has a unique solution* u *of class* $C^{1,2}$ *such that* u *and* $\nabla u\sigma$ *are bounded. Moreover, one has*

$$|u(t,x)| \leq (C+1)e^{D(T-t)} - 1 \quad \textit{and} \quad |\nabla u\sigma(t,x)| \leq \sqrt{n}\left(A + \frac{1-e^{-B(T-t)}}{B}G\right)Ee^{(B+F)(T-t)}$$

for all $(t,x) \in [0,T] \times \mathbb{R}^m$.

Proof. It follows from the assumptions by the mean value theorem that in case a), (C3)–(C4) are satisfied and in case b), (D3)–(D4) hold. So one obtains from Propositions 4.1 and 4.3 that in both cases, the BSDE (4.5) has a unique solution $(Y^{t,x}, Z^{t,x})$ in $\mathbb{S}_t^2(\mathbb{R}) \times \mathbb{H}_t^\infty(\mathbb{R}^n)$. Moreover,

$$|Z^{t,x}| \leq \sqrt{n}\left(A + \frac{1-e^{-B(T-t)}}{B}G\right)Ee^{(B+F)(T-t)},$$

and in case b), $|Y^{t,x}| \leq (C+1)e^{D(T-t)} - 1$. By modifying g for pairs (y,z) that are not attained by $(Y^{t,x}, Z^{t,x})$, one can assume that it is Lipschitz in (y,z). Then it follows from Theorem 3.2 of Pardoux and Peng [17] that $u(t,x) := Y_t^{t,x}$ defines a $C^{1,2}$ solution of the PDE (4.6). By Corollary 4.1 of El Karoui et al. [10], one has

$$|(\nabla u\sigma)(t,x)| = |Z_t^{t,x}| \leq \sqrt{n}\left(A + \frac{1-e^{-B(T-t)}}{B}G\right)Ee^{(B+F)(T-t)},$$

and in case b), $|u(t,x)| = |Y_t^{t,x}| \leq (C+1)e^{D(T-t)} - 1$.

Finally, let us prove uniqueness. In case a), if the PDE (4.6) has another solution v of class $C^{1,2}$ such that $\nabla v\sigma$ is bounded, it follows from Itô's lemma that $(\tilde{Y}^{t,x}_s, \tilde{Z}^{t,x}_s) = (v(s, X^{t,x}_s), (\nabla v\sigma)(s, X^{t,x}_s))$ is another solution of the BSDE (4.5). Boundedness of $\tilde{Z}^{t,x}$ implies that $\tilde{Y}^{t,x}$ is in $\mathbb{S}^2_t(\mathbb{R})$. By the uniqueness result of Propositions 4.1, one has $(Y^{t,x}, Z^{t,x}) = (\tilde{Y}^{t,x}, \tilde{Z}^{t,x})$, and therefore, $u = v$. In case b), uniqueness follows by the same argument. □

As a consequence of the results in this section, one obtains the following corollary for PDEs with initial conditions of the form

$$u_t = \triangle u + g(u, \nabla u), \quad u(0, x) = h(x), \tag{4.8}$$

where $u : [0, T] \times \mathbb{R}^n \to \mathbb{R}$.

Corollary 4.7. *Consider the following conditions:*

(i) *g and h satisfy* (C1)–(C2)*.*

(ii) *g and h satisfy* (D1)–(D2)*.*

(iii) *For every $L \in \mathbb{R}_+$ there exists a constant $\gamma_L \in \mathbb{R}$ such that $g(y', z) - g(y, z) \geq \gamma_L(y - y')$ for all $-L \leq y' \leq y \leq L$ and $z \in \mathbb{R}^n$.*

(iv) *g and h have bounded derivatives of first, second and third order on the set*

$$\left\{(x, y, z) \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n : |z| \leq \sqrt{n} A e^{BT}\right\}.$$

(v) *g and h have bounded derivatives of first, second and third order on the set*

$$\left\{(x, y, z) \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n : |y| \leq (C + 1)e^{DT} - 1, |z| \leq \sqrt{n} A e^{BT}\right\}.$$

Then the following hold:

a) *If* (i) *is satisfied, the PDE* (4.8) *has a viscosity solution u.*

b) *If* (ii) *is satisfied, the PDE* (4.8) *has a viscosity solution u satisfying $|u(t, x)| \leq (C + 1)e^{Dt} - 1$.*

c) *If* (ii) *and* (iii) *are satisfied, the PDE* (4.8) *has a unique bounded viscosity solution.*

d) *If* (i) *and* (iv) *are satisfied, the PDE* (4.8) *has a unique $C^{1,2}$-solution with bounded gradient ∇u, and $|\nabla u(t, x)| \leq \sqrt{n} A e^{Bt}$.*

e) *If* (ii) *and* (v) *are satisfied, the PDE* (4.8) *has a unique bounded $C^{1,2}$-solution with bounded gradient ∇u, and one has $|u(t, x)| \leq (C + 1)e^{Dt} - 1$ as well as $|\nabla u(t, x)| \leq \sqrt{n} A e^{Bt}$.*

Proof. Set $m = n$, $b \equiv 0$ and $\sigma \equiv \sqrt{2}\, Id$. Corollary 4.4 applied to $\tilde{g}(y, z) = g(y, z/\sqrt{2})$ yields that under (i) or (ii) the PDE with terminal condition,

$$v_t + \Delta v + g(v, \nabla v) = 0, \quad v(T, x) = h(x) \tag{4.9}$$

has a viscosity solution $v : [0, T] \times \mathbb{R}^n \to \mathbb{R}$. Moreover, if (ii) holds, one obtains from Proposition 4.3 that $|v(t, x)| \leq (C + 1)e^{D(T-t)} - 1$. It follows that under both conditions, (i) and (ii), $u(t, x) := v(T - t, x)$ is a viscosity solution of (4.8), which in case (ii) satisfies $u(t, x) \leq (C + 1)e^{Dt} - 1$. This shows a) and b). If (ii) and (iii) hold, one obtains from Corollary 4.5 that v is the unique bounded viscosity solution of (4.9). Therefore, u is the unique bounded viscosity solution of (4.8). This proves c). Finally, d) and e) follow from Corollary 4.6. □

Remark 4.8. In the special case $g(y,z) = \mu|z|^p$ the PDE (4.8) was studied by Amour and Ben-Artzi [1] as well as Gilding et al. [11]. Amour and Ben-Artzi [1] proved the existence and uniqueness of a classical solution for $\mu \neq 0$, $p > 1$ and h a bounded C^2 function with bounded derivatives of first and second order. Gilding et al. [11] proved the existence and uniqueness of a classical solution for $\mu = 1$, $p > 0$ and h a continuous bounded function. Equation (4.8) is more general, but for the existence of a viscosity solution we need g to be locally Lipschitz in z. To obtain a classical solution we have to assume that g and h are C^3.

5 BSDEs with random terminal times and PDEs with Dirichlet boundary conditions

5.1 BSDEs with random terminal times

Let $\tau \leq T$ be a stopping time and ξ an $\mathcal{F}_\tau$-measurable random variable.

Definition 5.1. *We say an $\mathbb{R} \times \mathbb{R}^n$-valued predictable process (Y, Z) solves the BSDE with random terminal time,*

$$Y_t = \xi + \int_{t\wedge\tau}^{\tau} f(s, Y_s, Z_s)ds - \int_{t\wedge\tau}^{\tau} Z_s dW_s, \tag{5.1}$$

if $\int_0^\tau (|f(t, Y_t, Z_t)|dt + |Z_t|^2)dt < \infty$, $Z_t = 0$ for $t > \tau$ and (5.1) is satisfied for all $0 \leq t \leq T$.

Suppose that for every $\omega \in \Omega$, the ODE

$$y_t(\omega) = \xi(\omega) - \int_{\tau(\omega)}^{t} f(s, \omega, y_s(\omega), 0)ds, \quad t \in [\tau(\omega), T], \tag{5.2}$$

has a unique solution $y(\omega)$, and set $\hat{\xi}(\omega) := y_T(\omega)$. Note that $1_{\{\tau\leq t\}} y_t$ is adapted, and in the special case $f(t, y, 0) = 0$, $t > \tau$, one has $\xi = \hat{\xi}$.

Proposition 5.2. *Assume $\hat{\xi}$ satisfies (A1) and f fulifills (A2)–(A4). Then the BSDE (5.1) has a unique solution (Y, Z) in $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$, and*

$$|Z_t^i| \leq \left(A_i + \int_t^T q_i(s) e^{-B(T-s)} ds \right) e^{B(T-t)} \quad dt \otimes d\mathbb{P}\text{-a.e.} \quad \text{for all } i = 1, ..., n. \tag{5.3}$$

Proof. If $\hat{\xi}$ satisfies (A1) and f fulfills (A2)–(A4), it follows from Theorem 1.2 that the BSDE

$$\hat{Y}_t = \hat{\xi} + \int_t^T f(s, \hat{Y}_s, \hat{Z}_s)ds - \int_t^T \hat{Z}_s dW_s$$

has a unique solution $(\hat{Y}, \hat{Z})$ in $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$, and $\hat{Z}$ satisfies the bound (5.3). Let

$$Q := \sqrt{\sum_{i=1}^n \left(A_i + \int_0^T q_i(t) e^{-B(T-t)} dt \right)^2} \, e^{BT},$$

and notice that $(\hat{Y}, \hat{Z})$ also solves the BSDE

$$\hat{Y}_t = \hat{\xi} + \int_t^T \hat{f}(s, \hat{Y}_s, \hat{Z}_s)ds - \int_t^T \hat{Z}_s dW_s, \tag{5.4}$$

where $\hat{f}$ is the 4-standard driver

$$\hat{f}(t,y,z) = \begin{cases} f(t,y,z) & \text{if } |z| \le Q \\ f(t,y,Qz/|z|) & \text{if } |z| > Q \end{cases}.$$

By Theorem 3.4 of Darling and Pardoux [6], the BSDE with random terminal time,

$$Y_t = \xi + \int_{t\wedge\tau}^{\tau} \hat{f}(s,Y_s,Z_s)ds - \int_{t\wedge\tau}^{\tau} Z_s dW_s, \tag{5.5}$$

has a unique solution (Y,Z) in $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^n)$. Now, note that the pair $(\tilde{Y},\tilde{Z})$ given by $\tilde{Y}_t := Y_t 1_{\{t\le\tau\}} + y_t 1_{\{\tau<t\}}$ and $\tilde{Z}_t := Z_t 1_{\{t\le\tau\}}$ is in $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^n)$ and solves the BSDE (5.4). But since (5.4) can only have one solution in $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^n)$, one has $(\tilde{Y},\tilde{Z}) = (\hat{Y},\hat{Z})$. In particular, (Y,Z) belongs to $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$, and Z satisfies the bound (5.3). It follows that (Y,Z) solves the BSDE (5.1).

Finally, if (Y',Z') is another solution in $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$ it must be equal to (Y,Z) since both solve the BSDE (5.5) for a 4-standard driver f' that coincides with f for $|z| \le Q'$, where $Q' \in \mathbb{R}_+$ is a bound on Z and Z'. □

Proposition 5.3. *If ξ is bounded by a constant $C \in \mathbb{R}_+$, $\hat{\xi}$ satisfies* (A1) *and f fulfills* (B2)–(B4) *with $R = (C+1)e^{2DT} - 1$ instead of $R = (C+1)e^{DT} - 1$, then the BSDE* (5.1) *has a unique solution (Y,Z) in $\mathbb{S}^\infty(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$, and*

$$|Y_t| \le (C+1)e^{D(T-t)} - 1 \quad \text{for all } t \in [0,T] \tag{5.6}$$

$$|Z_t^i| \le \left(A_i + \int_t^T q_i(s)e^{-B(T-s)}ds\right) e^{B(T-t)} \quad dt \otimes d\mathbb{P}\text{-a.e.} \quad \text{for all } i = 1,\dots,n. \tag{5.7}$$

Proof. By condition (B2), one has $|y_t(\omega)| \le C + \int_{\tau(\omega)}^t D(1+|y_s(\omega)|)ds$. So one obtains from Gronwall's lemma that $|\hat{\xi}| \le (C+1)e^{DT} - 1$. Now it follows from Corollary 1.4 by the same arguments as in the proof of Proposition 5.2 that the BSDE (5.1) has a unique solution (Y,Z) in $\mathbb{S}^\infty(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$ and the bound (5.7) is satisfied. To complete the proof, notice that since one has $Y_t = \xi$ and $Z_t = 0$ for $t > \tau$, (Y,Z) satisfies the BSDE

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)1_{\{s\le\tau\}}ds - \int_t^T Z_s dW_s.$$

So it follows from the comparison argument in the proof of Corollary 1.4 that (5.6) holds. □

5.2 Semilinear parabolic PDEs with Dirichlet boundary conditions

Let $\mathcal{O}$ be an open connected subset of $\mathbb{R}^m$. For every pair $(t,x) \in [0,T] \times \bar{\mathcal{O}}$, consider the SDE

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r)dW_r,$$

where b and σ fulfill the conditions (4.2)–(4.4). Define the stopping time

$$\tau^{t,x} := \inf\left\{s \ge t : X_s^{t,x} \notin \mathcal{O}\right\} \wedge T,$$

and consider the BSDE with random terminal time

$$Y_s^{t,x} = h(X_{\tau^{t,x}}^{t,x}) + \int_{s\wedge\tau^{t,x}}^{\tau^{t,x}} g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_{s\wedge\tau^{t,x}}^{\tau^{t,x}} Z_r^{t,x} dW_r, \quad t \le s \le T, \tag{5.8}$$

where $h : \bar{\mathcal{O}} \to \mathbb{R}$ and $g : [0,T] \times \bar{\mathcal{O}} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. Let $\bar{g} : [0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be an extension of g such that for every ω, the ODE

$$y_s^{t,x}(\omega) = h(X_{\tau^{t,x}}^{t,x}(\omega)) - \int_{\tau^{t,x}(\omega)}^{s} \bar{g}(r, X_r^{t,x}(\omega), y_r^{t,x}(\omega), 0)dr, \quad \tau^{t,x}(\omega) \le s \le T,$$

has a unique solution $y^{t,x}(\omega)$, and set $\hat{\xi}^{t,x}(\omega) := y_T^{t,x}(\omega)$. In the following two propositions we need $\bar{g}$ to satisfy the following condition:

(E) there exist constants $A_i \in \mathbb{R}_+$ such that for all $(t,x) \in [0,T] \times \bar{\mathcal{O}}$, $\hat{\xi}^{t,x} \in \mathbb{D}^{1,2}$ and $|D_r^i \hat{\xi}^{t,x}| \le A_i$ $dr \otimes d\mathbb{P}$-a.e. for all i.

Proposition 5.4. *Assume g has an extension $\bar{g} : [0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfying* (E) *and* (C2)–(C4) *with*

$$N = \sqrt{\sum_i \left(A_i + \frac{GEe^{FT}(1-e^{-BT})}{B}\right)^2} e^{BT}$$

instead of $N = \sqrt{n}\left(A + \frac{1-e^{BT}}{B}G\right)Ee^{(B+F)T}$. Then, for each pair $(t,x) \in [0,T] \times \bar{\mathcal{O}}$, the BSDE (5.8) *has a unique solution $(Y^{t,x}, Z^{t,x})$ in $\mathbb{S}_t^2(\mathbb{R}) \times \mathbb{H}_t^\infty(\mathbb{R}^n)$, and*

$$|Z_s^{t,x,i}| \le \left(A_i + \frac{GEe^{F(T-t)}(1-e^{-B(T-s)})}{B}\right) e^{B(T-s)} \quad ds \otimes d\mathbb{P}\text{-a.e.} \quad \text{for all } i = 1, \dots, n. \tag{5.9}$$

Proof. Fix (t,x), and set $\xi^{t,x} := h(X_{\tau^{t,x}}^{t,x})$. By assumption (E), $\hat{\xi}^{t,x}$ satisfies condition (A1), and it follows from the other assumptions like in the proof of Proposition 4.1 that $\bar{g}(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ fulfills (A2), (A3) with $q_i \equiv GEe^{F(T-t)}$ but without $\bar{g}(s, X_s^{t,x}, 0, 0) \in \mathbb{H}^4(\mathbb{R})$ and (A4) with a constant K. Now the proposition follows from Theorem 1.2 and Remark 1.3 like Proposition 5.4 followed from Theorem 1.2. $\square$

Proposition 5.5. *Assume h is bounded by a constant $C \in \mathbb{R}_+$ and g has an extension $\bar{g} : [0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfying* (E) *and* (D2)–(D4) *with*

$$N = \sqrt{\sum_i \left(A_i + \frac{GEe^{FT}(1-e^{-BT})}{B}\right)^2} e^{BT}$$

instead of $N = \sqrt{n}\left(A + \frac{1-e^{BT}}{B}G\right)Ee^{(B+F)T}$ and $R = (C+1)e^{2DT} - 1$ instead of $R = (C+1)e^{DT} - 1$. Then, for each $(t,x) \in [0,T] \times \bar{\mathcal{O}}$, the BSDE (5.8) *has a unique solution $(Y^{t,x}, Z^{t,x})$ in $\mathbb{S}_t^\infty(\mathbb{R}) \times \mathbb{H}_t^\infty(\mathbb{R}^n)$, and*

$$|Y_s^{t,x}| \le (C+1)e^{D(T-s)} - 1 \text{ for all } s \in [t,T]$$

$$|Z_s^{t,x,i}| \le \left(A_i + \frac{GEe^{F(T-t)}(1-e^{-B(T-s)})}{B}\right) e^{B(T-s)} \quad ds \otimes d\mathbb{P}\text{-a.e.} \quad \text{for all } i = 1, \dots, n.$$

Proof. The result follows from Corollary 1.4 like Proposition 5.4 follows from Theorem 1.2 and Remark 1.3. $\square$

An important assumption of Propositions 5.4 and 5.5 is that $\bar{g}$ satisfies condition (E). If, for instance, $\bar{g}(t,x,y,0) = 0$, then $\hat{\xi}^{t,x} = h(X^{t,x}_{\tau^{t,x}})$. So if h is Lipschitz continuous and $X^{t,x}_{\tau^{t,x}}$ is in $\mathbb{D}^{1,2}$ with bounded Malliavin derivative $D_r X^{t,x}_{\tau^{t,x}}$, it follows like in the proof of Proposition 4.1 that (E) holds.

Under appropriate assumptions, a solution to the BSDE (5.8) yields a solution to the following PDE with Dirichlet boundary conditions:

$$\begin{aligned} &u_t(t,x) + \mathcal{L}_{(t,x)} u(t,x) + g(t,x,u(t,x),(\nabla u\sigma)(t,x)) = 0 \quad \text{for } (t,x) \in [0,T) \times \mathcal{O} \\ &u(t,x) = h(x) \quad \text{for } (t,x) \in [0,T] \times \partial\mathcal{O} \text{ and } (t,x) \in \{T\} \times \mathcal{O}, \end{aligned} \tag{5.10}$$

where

$$\mathcal{L}_{(t,x)} := \frac{1}{2} \sum_{i,j} (\sigma\sigma^T)_{ij}(t) \partial_{x_i} \partial_{x_j} + \sum_i b_i(t,x) \partial_{x_i}.$$

The next result is a consequence of Theorem 2.2, Lemma 3.1 and Theorem 3.2 of Peng [19].

Theorem 5.6. (Peng, 1991) *Assume the following conditions hold:*

(F1) *$b : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ is in $C^{1,2}([0,T] \times \bar{\mathcal{O}})$, $\sigma : [0,T] \to \mathbb{R}^{m\times n}$ is in $C^1[0,T]$, and there exists a constant $\varepsilon > 0$ such that $\sum_{i,j} \left(\sigma\sigma^T\right)_{ij}(t) v_i v_j \geq \varepsilon |v|^2$ for all $(t,v) \in [0,T] \times \mathbb{R}^m$*

(F2) *$\mathcal{O}$ is bounded and $\partial\mathcal{O}$ is C^3*

(F3) *h is C^3 and $\mathcal{L}_{(t,x)} h(x) + g(T,x,h(x),\nabla h(x)\sigma(T)) = 0$ for $x \in \partial\mathcal{O}$*

(F4) *$g(t,x,y,z)$ is continuously differentiable in $(t,x,y,z) \in [0,T] \times \bar{\mathcal{O}} \times \mathbb{R} \times \mathbb{R}^n$ with bounded derivatives.*

Then the BSDE (5.8) *has a unique solution $(Y^{t,x}, Z^{t,x})$ in $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^n)$ and $u(t,x) := Y^{t,x}_t$ is the unique $C^{1,2}$-solution of the PDE* (5.10).

By applying Proposition 5.5, one can weaken condition (F4) in Theorem 5.6.

Corollary 5.7. *Assume* (F1)–(F3) *are satisfied, g is continuously differentiable in $(t,x,y,z) \in [0,T] \times \bar{\mathcal{O}} \times \mathbb{R} \times \mathbb{R}^n$ and the assumptions of Proposition 5.5 hold. Let $(Y^{t,x}, Z^{t,x})$ be the unique solution of the BSDE* (5.8) *in $\mathbb{S}^\infty_t(\mathbb{R}) \times \mathbb{H}^\infty_t(\mathbb{R}^n)$. Then $u(t,x) := Y^{t,x}_t$ is the unique $C^{1,2}$-solution of the PDE* (5.10), *and one has*

$$|u(t,x)| \leq (C+1)e^{D(T-t)} - 1, \quad |\nabla u(t,x)| \leq \frac{1}{\sqrt{\varepsilon}} \sqrt{\sum_i \left(A_i + \frac{GEe^{F(T-t)}(1-e^{-B(T-t)})}{B}\right)^2 e^{B(T-t)}}. \tag{5.11}$$

Proof. It follows from Proposition 5.5 that the BSDE (5.8) has a unique solution $(Y^{t,x}, Z^{t,x})$ in $\mathbb{S}^\infty_t(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$ with $|Y^{t,x}_s| \leq (C+1)e^{D(T-s)} - 1$ and

$$|Z^{t,x}_s| \leq \sqrt{\sum_i \left(A_i + \frac{GEe^{F(T-t)}(1-e^{-B(T-t)})}{B}\right)^2 e^{B(T-t)}} \quad ds \otimes d\mathbb{P}\text{-a.e.}$$

By modifying g for pairs (y,z) that are not attained by $(Y^{t,x}, Z^{t,x})$, one can assume that it has bounded derivatives. Then one obtains from Theorem 5.6 that $u(t,x) := Y^{t,x}_t$ is a $C^{1,2}$-solution of the PDE (5.10). It can be seen in the proof of Theorem 3.2 of Peng [19] that $Z^{t,x}_t = \nabla u(t,x)\sigma(t)$. So the bounds (5.11) follow from condition (F1).

If v is another $C^{1,2}$-solution of (5.10), v and ∇v are bounded. Moreover, it follows from Itô's formula that $\tilde{Y}^{t,x}_s := v(s \wedge \tau^{t,x}, X^{t,x}_{s\wedge\tau^{t,x}})$, $\tilde{Z}^{t,x}_s := \nabla v(s, X^{t,x}_{s\wedge\tau^{t,x}})\sigma(s)1_{\{s\le\tau\}}$ solve the BSDE (5.8). So one obtains from the uniqueness result of Proposition 5.5 that $u(t,x) = v(t,x)$. $\square$

6 Markovian BSDEs based on reflected SDEs and PDEs Neumann boundary conditions

In this whole section, $\mathcal{O} \subset \mathbb{R}^n$ is an open connected domain and $b : \bar{\mathcal{O}} \to \mathbb{R}^n$, $\sigma : \bar{\mathcal{O}} \to \mathbb{R}^{n\times n}$ are bounded Lipschitz functions. We assume that $\mathcal{O}$ satisfies the uniform exterior sphere condition and uniform interior cone condition introduced by Saisho [21]. They are defined as follows: For $y \in \partial\mathcal{O}$ and $r > 0$, define $\mathcal{N}_{y,r} := \{v \in \mathbb{R}^n : |v| = 1, B_r(y - rv) \cap \mathcal{O} = \emptyset\}$ and $\mathcal{N}_y := \cup_{r>0}\mathcal{N}_{y,r}$ where $B_r(y)$ denotes the ball around y with radius r.

Uniform exterior sphere condition
There exists a constant $r_0 > 0$ such that $\mathcal{N}_y = \mathcal{N}_{y,r_0} \neq \emptyset$ for all $y \in \partial\mathcal{O}$.

Uniform interior cone condition
There exist constants $\delta > 0$ and $\varepsilon \in [0,1)$ with the following property: for every $y \in \partial\mathcal{O}$, there exists a unit vector $v \in \mathbb{R}^n$ such that

$$\{z \in B_\delta(y) : \langle z - x, v\rangle \ge \varepsilon|z - x|\} \subset \bar{\mathcal{O}} \quad \text{for all } x \in B_\delta(y) \cap \partial\mathcal{O}.$$

6.1 Reflected SDEs and Markovian BSDEs

For every pair $(t,x) \in [0,T] \times \bar{\mathcal{O}}$ we define a diffusion $X^{t,x}$ that is reflected at the boundary of $\mathcal{O}$. Let $v(y) \in \mathcal{N}_y$ be a vector field on $\partial\mathcal{O}$. Note that if $\partial\mathcal{O}$ is smooth, then $v(y)$ is the unit inward normal vector at y. It is shown in Saisho [21] that for all (t,x), there exists a unique pair $(X^{t,x}, L^{t,x})$ of continuous adapted processes with values in $\bar{\mathcal{O}} \times \mathbb{R}_+$ such that for all $s \in [t,T]$,

$$\begin{aligned} X^{t,x}_s &= x + \int_t^s b(X^{t,x}_r)dr + \int_t^s \sigma(X^{t,x}_r)dW_r + \int_t^s v(X^{t,x}_r)dL^{t,x}_r \\ L^{t,x}_s &= \int_t^s 1_{\{X^{t,x}_r \in\partial\mathcal{O}\}}dL^{t,x}_r \quad \text{and} \quad L^{t,x} \text{ is nondecreasing.} \end{aligned} \tag{6.1}$$

Let $g : [0,T] \times \bar{\mathcal{O}} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $h : \bar{\mathcal{O}} \to \mathbb{R}$ be measurable functions and consider the BSDE

$$Y^{t,x}_s = h(X^{t,x}_T) + \int_s^T g(r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r)dr - \int_s^T Z^{t,x}_r dW_r, \quad t \le s \le T. \tag{6.2}$$

Proposition 6.1. *Assume there exists a constant $M \in \mathbb{R}_+$ such that for all $0 \le t \le s \le T$ and $x \in \bar{\mathcal{O}}$,*

$$X^{t,x}_s \in \mathbb{D}^{1,2} \quad \textit{and} \quad |D_r X^{t,x}_s| \le M \quad dr \otimes d\mathbb{P}\text{-a.e.} \tag{6.3}$$

If g and h satisfy (C1)–(C4) *with*

$$N = \sqrt{n}\left(A + \frac{1-e^{BT}}{B}G\right) Me^{BT} \quad \textit{instead of} \quad N = \sqrt{n}\left(A + \frac{1-e^{BT}}{B}G\right) Ee^{(B+F)T},$$

then (6.2) *has for all $(t,x) \in [0,T] \times \bar{\mathcal{O}}$ a unique solution $(Y^{t,x}, Z^{t,x}) \in \mathbb{S}^2_t(\mathbb{R}) \times \mathbb{H}^\infty_t(\mathbb{R}^n)$, and*

$$|Z^{t,x,i}_s| \leq \left(A + \frac{1-e^{-B(T-s)}}{B}G\right) Me^{B(T-s)} \quad ds \otimes d\mathbb{P}\text{-a.e.} \quad \text{for all } i = 1,\dots,n.$$

If g and h satisfy (D1)–(D4) *with*

$$N = \sqrt{n}\left(A + \frac{1-e^{BT}}{B}G\right) Me^{BT} \quad \textit{instead of} \quad N = \sqrt{n}\left(A + \frac{1-e^{BT}}{B}G\right) Ee^{(B+F)T},$$

then (6.2) *has a unique solution $(Y^{t,x}, Z^{t,x}) \in \mathbb{S}^\infty_t(\mathbb{R}) \times \mathbb{H}^\infty_t(\mathbb{R}^n)$, and*

$$\begin{aligned} |Y^{t,x}_s| &\leq (C+1)e^{D(T-s)} - 1 \quad \text{for all } s \in [t,T] \text{ a.s.} \\ |Z^{t,x,i}_s| &\leq \left(A + \frac{1-e^{-B(T-s)}}{B}G\right) Me^{B(T-s)} \quad ds \otimes d\mathbb{P}\text{-a.e.} \quad \text{for all } i = 1,\dots,n. \end{aligned}$$

Proof. If g and h satisfy (C1)–(C4), the proposition follows like Proposition 4.1, and if g and h fulfill (D1)–(D4), it follows like Proposition 4.3. □

(6.3) is a crucial assumption of Proposition 6.1. The following lemma gives a sufficient condition for it.

Lemma 6.2. *Assume that $\mathcal{O}$ is a convex polyhedron with nonempty interior in $\mathbb{R}^n$, $b = 0$, and $\sigma = c\,Id$ for a constant $c \in \mathbb{R}_+$. Then condition* (6.3) *holds.*

Proof. It follows from Theorems 2.1 and 2.2 in Dupuis and Ishii [9] that $X^{t,x}_s$ is Lipschitz continuous in W with constants $A_1,\dots,A_n$ independent of t, s and x. So the statement follows from Proposition 3.2. □

6.2 Semilinear Parabolic PDEs with Neumann boundary conditions

Assume that $\mathcal{O} \subset \mathbb{R}^n$ is bounded and there exists a function $w \in C^2(\mathbb{R}^n)$ with bounded derivatives of first and second order such that $\mathcal{O} = \{w > 0\}, \partial\mathcal{O} = \{w = 0\}, \mathbb{R}^n \setminus \bar{\mathcal{O}} = \{w < 0\}$, and $|\nabla w(x)| = 1$ for $x \in \partial\mathcal{O}$. Then $\mathcal{O}$ satisfies the uniform exterior sphere condition and uniform interior cone condition. So for all $(t,x) \in [0,T] \times \bar{\mathcal{O}}$, there exists a unique pair of continuous adapted processes $(X^{t,x}, L^{t,x})$ with values in $\bar{\mathcal{O}} \times \mathbb{R}_+$ such that

$$\begin{aligned} X^{t,x}_s &= x + \int_t^s b(X^{t,x}_r)dr + \int_t^s \sigma(X^{t,x}_r)dW_r + \int_t^s \nabla w(X^{t,x}_r)dL^{t,x}_r \\ L^{t,x}_s &= \int_t^s 1_{\{X^{t,x}_r \in \partial\mathcal{O}\}} dL^{t,x}_r \quad \text{and} \quad L^{t,x} \text{ is nondecreasing.} \end{aligned}$$

If the forward process is of this form, the Markovian BSDE (6.2) is related to the following PDE with Neumann boundary conditions for functions $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$:

$$\begin{aligned} & u_t(t,x) + \mathcal{L}_x u(t,x) + g\left(t,x,u(t,x),(\nabla u\sigma)(t,x)\right) = 0 \quad \text{for } (t,x) \in (0,T) \times \mathcal{O} \\ & \frac{\partial u}{\partial n}(t,x) = 0 \quad \text{for } (t,x) \in (0,T) \times \partial\mathcal{O} \quad \text{and} \quad u(T,x) = h(x) \quad \text{for } x \in \bar{\mathcal{O}}, \end{aligned} \tag{6.4}$$

where

$$\frac{\partial}{\partial n} := \sum_{i=1}^{n} \frac{\partial w}{\partial x_i}(x) \frac{\partial}{\partial x_i}, \quad \text{and} \quad \mathcal{L}_x := \frac{1}{2} \sum_{i,j} (\sigma\sigma^T)_{ij}(x) \partial_{x_i} \partial_{x_j} + \sum_i b_i(x) \partial_{x_i}.$$

Proposition 6.3. *Assume condition* (6.3) *holds and* g, h *satisfy* (D1)–(D4) *with*

$$N = \sqrt{n} \left(A + \frac{1 - e^{BT}}{B} G \right) M e^{BT} \quad \textit{instead of} \quad N = \sqrt{n} \left(A + \frac{1 - e^{BT}}{B} G \right) E e^{(B+F)T}.$$

Let $(Y^{t,x}, Z^{t,x})$ *be the solution of the BSDE* (6.2). *Then,* $u(t,x) := Y_t^{t,x}$ *is a viscosity solution of the PDE* (6.4) *satisfying* $|u(t,x)| \leq (C+1)e^{D(T-t)} - 1$ *for all* $(t,x) \in [0,T] \times \bar{\mathcal{O}}$.

Proof. One can assume that g is Lipschitz in (x,y,z) by modifying it for large (x,y,z). Then the results of Pardoux and Zhang [18] apply, and one obtains that $u(t,x) := Y_t^{t,x}$ is a viscosity solution of the PDE (6.4). By Proposition 6.1, it is bounded by $(C+1)e^{D(T-t)} - 1$. □

If one makes stronger assumptions on $\mathcal{O}, b, \sigma$ and g, the viscosity solution u of Proposition 6.3 is unique. We denote by $\mathcal{S}^n$ the set of all symmetric $n \times n$-matrices and define the function $F : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ by

$$F(t,x,y,v,S) := -\frac{1}{2} \sum_{i,j} (\sigma\sigma^T)_{ij}(x) S_{ij} - \sum_i b_i(x) v_i - g(T-t, x, y, v\sigma(x)).$$

Proposition 6.4. *Assume the boundary function* w *is* C^3 *with bounded derivatives of first, second and third order,* g *is continuous in* (t,x,y,z) *and the conditions of Proposition 6.3 hold. Moreover, suppose that for all* $L, L' \in \mathbb{R}_+$, *there exist a constant* $\gamma_L \in \mathbb{R}$ *and a function* $\delta_{L,L'} : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying* $\lim_{x \downarrow 0} \delta_{L,L'}(x) = 0$ *such that the following two conditions hold:*

(i) $g(t,x,y',v\sigma(x)) - g(t,x,y,v\sigma(x)) \geq \gamma_L(y - y')$ *for all* $(t,x) \in [0,T] \times \bar{\mathcal{O}}$, $-L \leq y' \leq y \leq L$ *and* $v \in \mathbb{R}^n$.

(ii)
$$F(t,x',y,v',S') - F(t,x,y,v,S) \leq \delta_{L,L'} \left(\eta + |x - x'|(1 + |v| \vee |v'|) + \frac{|x - x'|^2}{\varepsilon^2} \right)$$

for all $\eta, \varepsilon \in (0,1]$, $t \in [0,T]$, $x, x' \in \bar{\mathcal{O}}$, $|y| \leq L$, $v, v' \in \mathbb{R}^n$ *and* $S, S' \in \mathcal{S}^n$ *satisfying the following three properties:*

$$-\frac{L'}{\varepsilon^2} Id \leq \begin{pmatrix} S & 0 \\ 0 & -S' \end{pmatrix} \leq \frac{L'}{\varepsilon^2} \begin{pmatrix} Id & -Id \\ -Id & Id \end{pmatrix} + L'\eta Id$$
$$|v - v'| \leq L'\eta\varepsilon(1 + |v| \wedge |v'|)$$
$$|x - x'| \leq L'\eta\varepsilon.$$

Let $(Y^{t,x}, Z^{t,x})$ *be the solution of the BSDE* (6.2). *Then* $u(t,x) := Y_t^{t,x}$ *is the unique viscosity solution of the PDE* (6.4).

Proof. By Proposition 6.3, $u(t,x) := Y_t^{t,x}$ is a viscosity solution of (6.4). Uniqueness follows from Theorem 3.1 of Barles [2]. □

The unique viscosity solution of Proposition 6.4 is actually of class $C^{1,2}$ if one strengthens the assumptions.

Proposition 6.5. *Assume the conditions of Proposition 6.4 are satisfied and the following hold:*

(i) σ *is* $C^2(\bar{\mathcal{O}})$ *with bounded derivatives of first and second order and there exists a constant* $\varepsilon > 0$ *such that* $\sum_{i,j} \left(\sigma\sigma^T\right)_{ij}(x)v_i v_j \geq \varepsilon |v|^2$ *for all* $x, v \in \mathbb{R}^n$.

(ii) $b(x)v + g(t,x,y,v\sigma(x))$ *is continuously differentiable in* (t,x,y,v)

(iii) $h \equiv 0$.

Then the PDE (6.4) *has a unique* $C^{1,2}$*-solution* u*, and*

$$\begin{aligned} |u(t,x)| &\leq e^{D(T-t)} - 1 && (6.5)\\ |\nabla u(t,x)| &\leq \sqrt{\frac{n}{\varepsilon}} \left(A + \frac{1-e^{-B(T-t)}}{B} G\right) M e^{B(T-t)}. && (6.6) \end{aligned}$$

Proof. We can assume that g is Lipschitz in (x,y,z) by modifying it for large (x,y,x). Then it follows from Theorem V.7.4 of Ladyzenskaja et al. [15] that there exists a $C^{1,2}$ solution. So the unique viscosity solution u of Proposition 6.4 is $C^{1,2}$. From Pardoux and Zhang [18], we know that $Y^{t,x}_s = u(s, X^{t,x}_s)$. Since $h \equiv 0$, one obtains from Proposition 6.3 that u satisfies (6.5). Now fix $(t,x) \in (0,T) \times \mathcal{O}$ and let $\alpha > 0$ be a constant such that $\{y \in \mathbb{R} : |y - x| \leq \alpha\} \subset \mathcal{O}$. Define the stopping time $\tau^{t,x} := \inf\left\{s \geq t : |X^{t,x}_s - x| \geq \alpha\right\} \wedge (t+\alpha)$. Then $(Y^{t,x}_{s\wedge\tau^{t,x}}, Z^{t,x}_s 1_{\{s\leq\tau^{t,x}\}})$ and $(u(s\wedge\tau^{t,x}, X^{t,x}_{s\wedge\tau^{t,x}}), (\nabla u\sigma)(s, X^{t,x}_s)1_{\{s\leq\tau^{t,x}\}})$ are bounded solutions of the BSDE

$$\tilde{Y}_s = u(\tau^{t,x}, X^{t,x}_{\tau^{t,x}}) + \int_s^{t+\alpha} g(r, X^{t,x}_r, \tilde{Y}^{t,x}_r, \tilde{Z}^{t,x}_r) 1_{\{s\leq\tau^{t,x}\}} dr - \int_s^{t+\alpha} \tilde{Z}^{t,x}_r dW_r, \tag{6.7}$$

on $[t, t+\alpha]$. By modifying g for large (x,y,z), one can assume that it is Lipschitz in (x,y,z). Then (6.7) is a standard BSDE and has a unique solution. Therefore, one obtains from Proposition 6.1 that

$$|(\nabla u\sigma)(s, X^{t,x}_s)1_{\{s\leq\tau^{t,x}\}}| = |Z^{t,x}_s 1_{\{s\leq\tau^{t,x}\}}| \leq \sqrt{n}\left(A + \frac{1-e^{-B(T-s)}}{B} G\right) M e^{B(T-s)} \quad ds\otimes d\mathbb{P}\text{-a.e.}$$

on $[t, t+\alpha]$, and in particular,

$$|(\nabla u\sigma)(t,x)| \leq \sqrt{n}\left(A + \frac{1-e^{-B(T-t)}}{B} G\right) M e^{B(T-t)},$$

which by condition (i), gives the bound (6.6). $\square$

As a consequence of Propositions 6.1 and 6.3 one obtains the following result for PDEs of the form:

$$\begin{aligned} &u_t = u_{xx} + g(u, u_x) \quad \text{on } [0,T] \times (c,d)\\ &u_x = 0 \quad \text{on } \mathbb{R}_+ \times \{c,d\} \quad \text{and} \quad u(0,x) = h(x) \quad \text{for } x \in (c,d), \end{aligned} \tag{6.8}$$

where $u : [0,T] \times [c,d] \to \mathbb{R}$.

Corollary 6.6. *Assume h satisfies* (C1) *and g fulfills* (D2)*. Then* (6.8) *has a viscosity solution u satisfying*

$$|u(t,x)| \leq \left(\sup_{c<x<d} |h(x)| + 1 \right) e^{Dt} - 1, \quad (t,x) \in [0,T] \times [c,d].$$

Moreover, if g is continuous in y, for every $L \in \mathbb{R}_+$, there exists a constant $\gamma_L \in \mathbb{R}$ such that for all $-L \leq y' \leq y \leq L$ and $z \in \mathbb{R}^n$, one has

$$g(y', z) - g(y, z) \geq \gamma_L (y - y') \tag{6.9}$$

and $F(t, x, y, v, S) = \sum_{i,j} S_{ij} - g(y, v)$ satisfies condition (ii) *of Proposition 6.4, then u is the unique viscosity solution. If in addition, $h \equiv 0$ and g is C^1, then u is $C^{1,2}$ and satisfies*

$$|u_x(t,x)| \leq 3Ae^{Bt} \quad \textit{for all } (t,x) \in [0,T] \times [c,d].$$

Proof. Set $b \equiv 0$, $\sigma \equiv \sqrt{2}\, Id$ and $\tilde{g}(y,z) := g(y, z/\sqrt{2})$. Since h is Lipschitz continuous and $[c,d]$ is compact, h is bounded. Therefore, $\tilde{g}$ and h satisfy (D1)–(D4) with $C = \sup_{c<x<d} |h(x)|$ and $G = H = 0$. So one obtains from Proposition 6.1 and Lemma 6.2 that the BSDE (6.2) has a unique solution $(Y^{t,x}, Z^{t,x})$ in $\mathbb{S}^\infty_t(\mathbb{R}) \times \mathbb{H}^\infty_t(\mathbb{R}^n)$ with $|Y^{t,x}_s| \leq (C+1)e^{D(T-s)} - 1$. It can be seen from Theorems 2.1 and 2.2 of Dupuis and Ishii [9] together with Proposition 3.2 that condition (6.3) is satisfied with $M = 3\sqrt{2}$. Therefore, Proposition 6.1 yields $|Z^{t,x}_s| \leq 3\sqrt{2}Ae^{B(T-s)}$. By Proposition 6.3, $v(t,x) := Y^{t,x}_t$ is a viscosity solution of the PDE

$$\begin{aligned} &v_t + v_{xx} + g(v, v_x) = 0 \quad \text{on } [0,T] \times (c,d) \\ &v_x = 0 \quad \text{on } [0,T] \times \{c,d\} \quad \text{and} \quad v(T,x) = h(x) \quad \text{for } x \in (c,d), \end{aligned}$$

satisfying $|v(t,x)| \leq (C+1)e^{D(T-t)} - 1$. So $u(t,x) := v(T-t,x)$ is a viscosity solution of (6.8) with $|u(t,x)| \leq (C+1)e^{Dt} - 1$. If g is continuous in y, (6.9) holds and $F(t,x,y,v,S) = \sum_{i,j} S_{ij} - g(y,v)$ fulfills condition (ii) of Proposition 6.4, then the conditions of Proposition 6.4 are satisfied. So u is the unique viscosity solution. If in addition, $h \equiv 0$ and g is of class C^1, one obtains from Proposition 6.5 that u is of class $C^{1,2}$ and $|u_x(t,x)| \leq 3Ae^{Bt}$. □

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